Deriving Moments of Inertia

Deriving moment of inertia from kinetic energy

$$
\begin{aligned}
T &= \frac{1}{2}mv^2 = \frac{1}{2}\int \left(r\dot{\theta}\right)^2 dm \\
&= \frac{1}{2}\left[\int r^2 dm\right]\dot{\theta}^2 \\
&= \frac{1}{2}I\omega^2 \\
I &= \int r^2 dm
\end{aligned}
$$

Hoop of radius R

$$
\begin{aligned}
I &= \int R^2 dm \\
&= MR^2
\end{aligned}
$$

Disc of radius R $\left(\sigma = M/\pi R^2\right)$

$$
\begin{aligned}
I &= \int r^2 dm \\
&= \int r^2 \cdot \sigma r dr d\theta \\
&= \sigma \int_0^R \int_0^{2\pi} r^3 d\theta dr \\
&= \sigma \cdot 2\pi \int_0^R r^3 dr \\
&= \sigma 2\pi \cdot \frac{R^4}{4} \\
&= \frac{M}{\pi R^2} \cdot \frac{\pi R^4}{2} \\
&= \frac{MR^2}{2}
\end{aligned}
$$

Rod of length L from center ($\lambda = M/L$)

$$\begin{aligned} I &= \int r^2 dm \\ &= \int_{-L/2}^{L/2} x^2 \lambda dx \\ &= \frac{\lambda}{3}\left[\left(\frac{L}{2}\right)^3 + \left(\frac{L}{2}\right)^3\right] \\ &= \frac{M}{L} \cdot \frac{L^3}{12} \\ &= \frac{ML^2}{12} \end{aligned}$$

Rod of length L from end ($\lambda = M/L$)

$$\begin{aligned} I &= \int r^2 dm \\ &= \int_0^L x^2 \lambda dx \\ &= \lambda \frac{L^3}{3} \\ &= \frac{M}{L} \cdot \frac{L^3}{3} \\ &= \frac{ML^2}{3} \end{aligned}$$

Rod of length L through vertical

Equivalent to a disc.

Rectangle of sides a and b from corner $(\sigma = M/ab)$

$$\begin{aligned}
I &= \int r^2 dm \\
&= \int_0^a \int_0^b \left(x^2 + y^2\right) \sigma dx dy \\
&= \sigma \int_0^a \left(x^2 b + \frac{b^3}{3}\right) dx \\
&= \sigma \left(\frac{a^3}{3} b + \frac{b^3}{3} a\right) \\
&= \frac{M}{ab} \frac{a^3 b + a b^3}{3} \\
&= M \frac{a^2 + b^2}{3}
\end{aligned}$$

Rectangle of sides a and b from center $(\sigma = M/ab)$

$$\begin{aligned}
I &= \int r^2 dm \\
&= \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left(x^2 + y^2\right) \sigma dx dy \\
&= \sigma \int_{-a/2}^{a/2} \left(x^2 b + \frac{b^3}{12}\right) dx \\
&= \sigma \left(\frac{a^3}{12} b + \frac{b^3}{12} a\right) \\
&= \frac{M}{ab} \cdot \frac{a^3 b + a b^3}{12} \\
&= M \frac{a^2 + b^2}{12}
\end{aligned}$$