Seth's Random Physics Problem

Question

If you had an object accelerating upwards for a short period of time T - i.e. a rocket - the relationship between the magnitude of the initial acceleration A and the height the rocket reaches $y_{\max}$ would be quadratic, not linear, since it would have a greater upward velocity for a longer period of time, right?

Answer

The statement "it has a greater upward velocity for a longer period of time" implies neither a linear nor quadratic relationship. A bigger initial velocity will result in a bigger $y_{\max}$ for either relationship. To figure out which it is, let's derive the maximum height of a projectile.

As soon as the rocket is no longer accelerating at A, it will be decelerating with gravity.

$$\begin{aligned}\frac{d^2y}{dt^2} &= -g \\ \frac{dy}{dt} &= -gt + v_0 \\ y(t) &= -\frac{1}{2}gt^2 + v_0 t + y_0\end{aligned}$$

We can let y_0 be zero. Then solve for the total flight time, i.e. find when the projectile hits the ground again ($y = 0$).

$$
\begin{aligned}
y(t_f) = 0 &= -\frac{1}{2}gt_f^2 + v_0 t_f \\
&= -\frac{1}{2}gt_f + v_0 \\
t_f &= \frac{2v_0}{g}
\end{aligned}
$$

The time when y is maximum is half of the total flight time. Plug in this time to find the maximum height.

$$
\begin{aligned}
y_{\max} = y\left(\frac{v_0}{g}\right) &= -\frac{1}{2}g\left(\frac{v_0^2}{g^2}\right) + v_0\left(\frac{v_0}{g}\right) \\
&= -\frac{1}{2}\frac{v_0^2}{g} + \frac{v_0^2}{g} \\
&= \frac{v_0^2}{2g}
\end{aligned}
$$

Remember that the initial velocity v_0 is not an arbitrary constant in this case; it is a function of A and T.

$$
v_0 = AT
$$

Plug it in:

$$
\begin{aligned}
y_{\max} &= \frac{(AT)^2}{2g} \\
&= \frac{A^2T^2}{2g}
\end{aligned}
$$

This is definitely a quadratic relationship.