# Solution to that one problem I could do in Physics 311

## Thornton & Marion, *Classical Dynamics*: 2-19

If a projectile moves such that its distance from the point of projection is always increasing, find the maximum angle above the horizontal with which the particle could have been projected. (Assume no air resistance.)

### Solution

If the distance $s$ is always increasing, the time derivative of $s$ must always be greater than zero.

$$\frac{ds}{dt} > 0 \tag{1}$$

The distance is given by

$$s(x, y) = \sqrt{x^2 + y^2} \tag{2}$$

Then

$$\frac{ds}{dt} = \frac{\partial s}{\partial x}\frac{dx}{dt} + \frac{\partial s}{\partial y}\frac{dy}{dt} + \frac{\partial s}{\partial t} \tag{3}$$

$$\frac{\partial s}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \qquad \frac{\partial s}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} \tag{4}$$

So the critical condition is

$$0 = x\dot{x} + y\dot{y} \tag{5}$$

From Newton's equations,

$$x(t) = v_0 t \cos\alpha \qquad y(t) = v_0 t \sin\alpha - \frac{1}{2}gt^2 \tag{6}$$

$$\dot{x}(t) = v_0 \cos\alpha \qquad \dot{y}(t) = v_0 \sin\alpha - gt \tag{7}$$

Plugging these in and expanding, we get

$$0 = v_0^2 t(\cos^2\alpha + \sin^2\alpha) + \frac{1}{2}g^2t^3 - \frac{3}{2}gt^2 v_0 \sin\alpha \tag{8}$$

After simplification,

$$\sin\alpha = \frac{gt}{3v_0} + \frac{2v_0}{3gt} \tag{9}$$

Maximize the right hand side to get the time when the distance is closest to decreasing.

$$\frac{d}{dt}\left(\frac{gt}{3v_0} + \frac{2v_0}{3gt}\right) = 0 \tag{10}$$

$$\frac{g}{3v_0} - \frac{2v_0}{3gt^2} = 0 \tag{11}$$

We find that the critical time is

$$t = \sqrt{2}\frac{v_0}{g} \tag{12}$$

Plug this in to (9) and solve for $\alpha$:

$$\alpha = \arcsin\left(\frac{2\sqrt{2}}{3}\right) \approx 70.5 \tag{13}$$